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Generalized Taub–NUT metrics and Killing–Yano tensors

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Abstract. The necessary condition that a Stäckel–Killing tensor of valence two should be the contracted product of a Killing–Yano tensor of valence two with itself is rederived for a Riemannian manifold. This condition is applied to the generalized Euclidean Taub–NUT metrics which admit a Kepler-type symmetry. It is shown that, in general, the Stäckel–Killing tensors involved in the Runge–Lenz vector cannot be expressed as a product of Killing–Yano tensors. The only exception is the original Taub–NUT metric.

1. Introduction

The Euclidean Taub–NUT metric is involved in many modern studies in physics. Hawking [1] has suggested that the Euclidean Taub–NUT metric might give rise to the gravitational analogue of the Yang–Mills instanton. In this case Einstein’s equations are satisfied with zero cosmological constant and the manifold is \mathbb{R}^4 with a boundary which is a twisted 3-sphere S^3 possessing a distorted metric. The Kaluza–Klein monopole was obtained by embedding the Taub–NUT gravitational instanton into five-dimensional Kaluza–Klein theory. On the other hand, in the long-distance limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of this space [2, 3].

From the symmetry viewpoint, the geodesic motion in Taub–NUT space admits a ‘hidden’ symmetry of the Kepler-type if a cyclic variable is removed [4–7]. In general the ‘hidden’ symmetries of the manifold manifest themselves as Stäckel–Killing tensors of valence $r > 1$ [8]. The conserved quantities along geodesics are homogeneous functions in momentum p_μ of degree r , and which commute with the Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \quad (1)$$

in the sense of Poisson brackets.

In the Taub–NUT geometry there are four Killing–Yano tensors [9]. Three of these are complex structure realizing the quaternionic algebra and the Taub–NUT manifold is hyper-Kähler [5]. In addition to these three vector-like Killing–Yano tensors, there is a scalar one which has a non-vanishing field strength and exists by virtue of the metric being of type D .

For the geodesic motions in the Taub–NUT space, the conserved vector analogous to the Runge–Lenz vector of the Kepler-type problem is quadratic in 4-velocities, its components are Stäckel–Killing tensors and they can be expressed as symmetrized products of Killing–Yano tensors [5, 10–12].

The Killing–Yano tensors play an important role in the models for relativistic spin- $\frac{1}{2}$ particles involving anticommuting vectorial degrees of freedom, usually called the spinning particles [13–16]. The configuration space of spinning particles (spinning space) is an extension of an ordinary Riemannian manifold, parametrized by local coordinates $\{x^\mu\}$, to a graded manifold parametrized by local coordinates $\{x^\mu, \psi^\mu\}$, with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anticommuting). In the spinning case the generalized Killing equations are more involved and new procedures have been conceived [12, 15]. In particular, if the Killing tensors can be written in terms of Killing–Yano tensors (and that is the case of the Taub–NUT space), the generalized Killing equations can be solved explicitly in a simple, closed form.

Iwai and Katayama [17–20] extended the Taub–NUT metric so that it still admits a Kepler-type symmetry. This class of metrics, of course, includes the original Taub–NUT metric.

The aim of this paper is to investigate whether the Stäckel–Killing tensors involved in the conserved Runge–Lenz vector of the extended Taub–NUT metrics can also be expressed in terms of Killing–Yano tensors.

The relationship between Killing tensors and Killing–Yano tensors has been investigated with regard to the Lorentzian geometry used in general relativity [21, 22]. In the next section we re-examine the condition that a Killing tensor of valence two should be the contracted product of a Killing–Yano tensor of valence two with itself. The procedure is quite simple and is specific for the Riemannian geometry appropriate to Euclidean Taub–NUT metrics.

In section 3 we show that, in general, the Killing tensors involved in the Runge–Lenz vector cannot be expressed as a product of Killing–Yano tensors. The only exception is the original Taub–NUT metric.

Our comments and concluding remarks are presented in section 4.

2. The relationship between Killing tensors and Killing–Yano tensors

We consider a four-dimensional Riemannian manifold M and a metric $g_{\mu\nu}(x)$ on M in local coordinates x^μ . We write the metric in terms of the local orthonormal vierbein frame e^a_μ

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = \sum_{a=0,1,2,3} (e^a)^2 \quad (2)$$

where $e^a = e^a_\mu dx^\mu$. Greek indices μ, ν, \dots are raised and lowered with $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$, while Latin indices a, b, \dots are raised and lowered by the flat metric δ_{ab} , $a, b = 0, 1, 2, 3$. Vierbeins and inverse vierbeins convert between Latin and Greek indices when necessary.

The following two generalization of the Killing vector equation have become of interest in physics [22].

- (a) A tensor $f_{\mu_1 \dots \mu_r}$ is called a Killing–Yano tensor of valence r if it is totally antisymmetric and it satisfies the equation

$$f_{\mu_1 \dots (\mu_r; \lambda)} = 0. \quad (3)$$

- (b) A symmetric tensor field $K_{\mu_1 \dots \mu_r}$ is called a Stäckel–Killing tensor of valence r iff

$$K_{(\mu_1 \dots \mu_r; \lambda)} = 0. \quad (4)$$

Let Λ^2 be the space of 2-forms $\Lambda^2 := \Lambda^2 T^*(\mathbb{R}^4 - \{0\})$. We define self-dual and anti-self-dual bases for Λ^2 using the vierbein 1-forms e^a [23]:

$$\text{basis of } \Lambda^2_\pm = \begin{cases} \lambda^1_\pm = e^0 \wedge e^1 \pm e^2 \wedge e^3 \\ \lambda^2_\pm = e^0 \wedge e^2 \pm e^3 \wedge e^1 \\ \lambda^3_\pm = e^0 \wedge e^3 \pm e^1 \wedge e^2. \end{cases} \quad * \lambda^i_\pm = \pm \lambda^i_\pm \quad (5)$$

Let f be a Killing–Yano tensor of valence two and $*f$ its dual. The symmetric combination of f and $*f$ is a self-dual 2-form

$$f + *f = \sum_{i=1,2,3} y_i \lambda_+^i \tag{6}$$

while their difference is an antiself-dual 2-form

$$f - *f = \sum_{i=1,2,3} z_i \lambda_-^i. \tag{7}$$

An explicit evaluation shows that

$$(f + *f)^2 = - \sum_{i=1,2,3} (y_i)^2 \cdot \mathbb{1} \tag{8}$$

$$(f - *f)^2 = - \sum_{i=1,2,3} (z_i)^2 \cdot \mathbb{1} \tag{9}$$

where $\mathbb{1}$ is a 4×4 identity matrix.

Let us suppose that a Stäckel–Killing tensor $K_{\mu\nu}$ can be written as the contracted product of a Killing–Yano tensor $f_{\mu\nu}$ with itself:

$$K_{\mu\nu} = f_{\mu\lambda} \cdot f^\lambda{}_\nu = (f^2)_{\mu\nu} \quad \mu, \nu = 0, 1, 2, 3. \tag{10}$$

We infer from the last equations that

$$K + \frac{1}{16} \left[\sum_i (y_i^2 - z_i^2) \right]^2 K^{-1} + \frac{1}{2} \sum_i (y_i^2 + z_i^2) \cdot \mathbb{1} = 0. \tag{11}$$

On the other hand, the Killing tensor K is symmetric and it can be diagonalized with the aid of an orthogonal matrix. Its eigenvalues satisfy an equation of the second degree:

$$\lambda_\alpha^2 + \frac{1}{2} \sum_i (y_i^2 + z_i^2) \lambda_\alpha + \frac{1}{16} \left[\sum_i (y_i^2 - z_i^2) \right]^2 = 0 \tag{12}$$

with at most two distinct roots.

In conclusion, a Stäckel–Killing tensor K which can be written as the square of a Killing–Yano tensor has at the most two distinct eigenvalues.

3. Generalized Taub–NUT metrics

For a special choice of coordinates the generalized Euclidean Taub–NUT metric considered by Iwai and Katayama [17–20] takes the form

$$ds_G^2 = f(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2] + g(r)[d\chi + \cos \theta d\varphi]^2 \tag{13}$$

where $r > 0$ is the radial coordinate of $\mathbb{R}^4 - \{0\}$, the angle variables (θ, φ, χ) , $(0 \leq \theta < \pi, 0 \leq \varphi < 2\pi, 0 \leq \chi < 4\pi)$ parametrize the unit sphere S^3 , and $f(r)$ and $g(r)$ are arbitrary functions of r .

We decompose the metric (13) into the orthogonal vierbein basis:

$$\begin{aligned} e^0 &= g(r)^{\frac{1}{2}} (d\chi + \cos \theta d\varphi) \\ e^1 &= r f(r)^{\frac{1}{2}} (\sin \chi d\theta - \sin \theta \cos \chi d\varphi) \\ e^2 &= r f(r)^{\frac{1}{2}} (-\cos \chi d\theta - \sin \theta \sin \chi d\varphi) \\ e^3 &= f(r)^{\frac{1}{2}} dr. \end{aligned} \tag{14}$$

Spaces with a metric of the above form have an isometry group $SU(2) \times U(1)$. The four Killing vectors are

$$D_A = R_A^\mu \partial_\mu \quad A = 0, 1, 2, 3 \tag{15}$$

where

$$\begin{aligned} D_0 &= \frac{\partial}{\partial \chi} \\ D_1 &= -\sin \varphi \frac{\partial}{\partial \theta} - \cos \varphi \cot \theta \frac{\partial}{\partial \varphi} + \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \chi} \\ D_2 &= \cos \varphi \frac{\partial}{\partial \theta} - \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} + \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \chi} \\ D_3 &= \frac{\partial}{\partial \varphi}. \end{aligned} \tag{16}$$

D_0 which generates the $U(1)$ of χ translations, commutes with the other Killing vectors. In turn, the remaining three vectors, corresponding to the invariance of the metric (13) under spatial rotations ($A = 1, 2, 3$), obey an $SU(2)$ algebra with

$$[D_1, D_2] = -D_3 \quad \text{etc.} \tag{17}$$

Let us consider geodesic flows of the generalized Taub–NUT metric which has the Lagrangian L on the tangent bundle $T(\mathbb{R}^4 - \{0\})$

$$L = \frac{1}{2} f(r) [\dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2)] + \frac{1}{2} g(r) (\dot{\chi} + \cos \theta \dot{\varphi})^2 \tag{18}$$

where $(\dot{r}, \dot{\theta}, \dot{\varphi}, \dot{\chi}, r, \theta, \varphi, \chi)$ denote coordinates in the tangent bundle. Since χ is a cyclic variable

$$q = g(r) (\dot{\theta} + \cos \theta \dot{\varphi}) \tag{19}$$

is a conserved quantity. This is known in the literature as the ‘relative electric charge’.

Taking into account this cyclic variable, the dynamical system for the geodesic flow on $T(\mathbb{R}^4 - \{0\})$ can be reduced to a system on $T(\mathbb{R}^3 - \{0\})$. The reduced system admits manifest rotational invariance, and hence has a conserved angular momentum

$$\vec{J} = \vec{r} \times \vec{p} + q \frac{\vec{r}}{r} \tag{20}$$

where \vec{r} denotes the 3-vector $\vec{r} = (r, \theta, \varphi)$ and $\vec{p} = f(r) \dot{\vec{r}}$ is the mechanical momentum.

If $f(r)$ and $g(r)$ are taken to be

$$f(r) = \frac{4m + r}{r} \quad g(r) = \frac{16m^2 r}{4m + r} \tag{21}$$

the metric ds_G^2 becomes the original Euclidean Taub–NUT metric. As observed in [5], the Taub–NUT geometry also possesses four Killing–Yano tensors of valence two. The first three are rather special: they are covariantly constant (with vanishing field strength)

$$\begin{aligned} f_i &= 8m(d\chi + \cos \theta d\varphi) \wedge dx_i - \epsilon_{ijk} \left(1 + \frac{4m}{r}\right) dx_j \wedge dx_k \\ D_\mu f_{i\lambda}^\nu &= 0 \quad i, j, k = 1, 2, 3. \end{aligned} \tag{22}$$

They are mutually anticommuting and square the minus unity:

$$f_i f_j + f_j f_i = -2\delta_{ij}. \tag{23}$$

Thus they are complex structures realizing the quaternion algebra. Indeed, the Taub–NUT manifold defined by (13) and (21) is hyper-Kähler.

In addition to the above vector-like Killing–Yano tensors there also is a scalar one

$$f_Y = 8m(d\chi + \cos\theta d\varphi) \wedge dr + 4r(r + 2m)\left(1 + \frac{r}{4m}\right) \sin\theta d\theta \wedge d\varphi \tag{24}$$

which has a non-vanishing component of the field strength

$$f_{Yr\theta;\varphi} = 2\left(1 + \frac{r}{4m}\right)r \sin\theta. \tag{25}$$

In the original Taub–NUT case there is a conserved vector analogous to the Runge–Lenz vector of the Kepler-type problem:

$$\vec{K} = \frac{1}{2}\vec{K}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \vec{p} \times \vec{j} + \left(\frac{q^2}{4m} - 4mE\right)\frac{\vec{r}}{r} \tag{26}$$

where the conserved energy E , from equation (1), is

$$E = \frac{\vec{p}^2}{2f(r)} + \frac{q^2}{2g(r)}. \tag{27}$$

The components $K_{i\mu\nu}$ involved with the Runge–Lenz-type vector (26) are Killing tensors and they can be expressed as symmetrized products of the Killing–Yano tensors f_i (22) and f_Y (24) [11, 12]:

$$K_{i\mu\nu} - \frac{1}{8m}(R_{0\mu}R_{i\nu} + R_{0\nu}R_{i\mu}) = m(f_{Y\mu\lambda}f_{i\lambda\nu} + f_{Y\nu\lambda}f_{i\lambda\mu}). \tag{28}$$

Returning to the generalized Taub–NUT metric, in analogy with equation (26), Iwai and Katayama [17–20] assumed that in addition to the angular momentum vector there exist a conserved vector \vec{S} of the following form:

$$\vec{S} = \vec{p} \times \vec{J} + \kappa\frac{\vec{r}}{r} \tag{29}$$

with an unknown constant κ .

It was found that the metric (13) still admits a Kepler-type symmetry (29) if the functions $f(r)$ and $g(r)$ take, respectively, the form

$$f(r) = \frac{a + br}{r} \quad g(r) = \frac{ar + br^2}{1 + cr + dr^2} \tag{30}$$

where a, b, c, d are constants. The constant κ involved in the Runge–Lenz vector (29) is

$$\kappa = -aE + \frac{1}{2}cq^2. \tag{31}$$

If $ab > 0$ and $c^2 - 4d < 0$ or $c > 0, d > 0$, no singularity of the metric appears in $\mathbb{R}^4 - \{0\}$. On the other hand, if $ab < 0$ a manifest singularity appears at $r = -a/b$ [18].

It is straightforward to verify that the components of the vector \vec{S} are Stäckel–Killing tensors in the extended Taub–NUT space (13) with the function $f(r)$ and $g(r)$ given by (30). Moreover, the Poisson brackets between the components of \vec{J} and \vec{S} are [17]

$$\begin{aligned}\{J_i, J_j\} &= \epsilon_{ijk} J_k \\ \{J_i, S_j\} &= \epsilon_{ijk} S_k \\ \{S_i, S_j\} &= (d q^2 - 2 b E) \epsilon_{ijk} J_k\end{aligned}\quad (32)$$

as is expected from the same relations known for the original Taub–NUT metric.

Our task is to investigate whether the components of the Runge–Lenz vector (29) can be the contracted product of Killing–Yano tensors of valence two. For the model of equation (28) from the original Taub–NUT case it is not required that a component S_i of the Runge–Lenz vector (29) be directly expressed as a symmetrized product of Killing–Yano tensors. Taking into account that \vec{S} transforms as a vector under rotations generated by \vec{J} , equation (32), the components $S_{i\mu\nu}$ can be combined with trivial Stäckel–Killing tensors of the form $(R_{0\mu} R_{i\nu} + R_{0\nu} R_{i\mu})$ to obtain the appropriate tensor which has to be decomposed into a product of Killing–Yano tensors.

In order to use the results from the previous section, we shall write the symmetrized product of two different Killing–Yano tensors f' and f'' as a contracted product of $f' + f''$ with itself, extracting adequately the contribution of f'^2 and f''^2 . Since the generalized Taub–NUT space (13) does not admit any other non-trivial Stäckel–Killing tensor besides the metric $g_{\mu\nu}$ and the components $S_{i\mu\nu}$ of (29), f'^2 and f''^2 should be connected with the scalar conserved quantities E , \vec{J}^2 , q^2 through the tensors $g_{\mu\nu}$, $\sum_{A=1,2,3} R_{A\mu} R_{Av}$ and $R_{0\mu} R_{0\nu}$.

In conclusion we shall consider a general linear combination between a component S_i of the Runge–Lenz vector (29) and symmetrized pairs of Killing vectors of the form

$$S_{iab} + \alpha_1 \sum_{A=1}^3 R_{Aa} R_{Ab} + \alpha_2 R_{0a} R_{0b} + \alpha_3 (R_{0a} R_{ib} + R_{ia} R_{0b}) \quad (33)$$

where α_i are constants. We are looking for the conditions for the above tensor to be the contracted product of a Killing–Yano tensor with itself. For this purpose we evaluate the eigenvalues of the matrix (33) and we find that it has at most two distinct eigenvalues if and only if

$$\begin{aligned}\alpha_1 + \alpha_2 &= 0 \\ \alpha_3 &= -\frac{1}{4}c \\ d &= \frac{1}{4}c^2.\end{aligned}\quad (34)$$

For example, if the above conditions are satisfied, the eigenvalues of the matrix (33) for the third component S_3 of the Runge–Lenz vector (29) are

$$\lambda_1 = \frac{1}{2}(br \cos \theta + (a + br)(r\alpha_1 + \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta})) \quad (35)$$

with the eigenvectors

$$\begin{aligned}\{ \tan \chi, (r\alpha_1 + \cos \theta + \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta}) \operatorname{cosec} \theta \sec \chi, 0, 1 \} \\ \{ (r\alpha_1 + \cos \theta - \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta}) \operatorname{cosec} \theta \sec \chi, -\tan \chi, 1, 0 \}\end{aligned}\quad (36)$$

and

$$\lambda_2 = \frac{1}{2}(br \cos \theta + (a + br)(r\alpha_1 - \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta})) \tag{37}$$

with the eigenvectors

$$\begin{aligned} &\{ \tan \chi, (r\alpha_1 + \cos \theta - \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta}) \operatorname{cosec} \theta \sec \chi, 0, 1 \} \\ &\{ (r\alpha_1 + \cos \theta + \sqrt{1 + r^2\alpha_1^2 + 2r\alpha_1 \cos \theta}) \operatorname{cosec} \theta \sec \chi, -\tan \chi, 1, 0 \}. \end{aligned} \tag{38}$$

Hence the constants involved in the functions f, g are constrained, restricting their expressions accordingly. It is worth mentioning that if relation (34) between constants c and d is satisfied, the metric is conformally self-dual or antiself-dual depending upon the sign of the quantity $2 + cr$ [18]. More precisely, for the Weyl curvature tensor

$$C^i_{jkl} = R^i_{jkl} - \frac{1}{2}(\delta^i_k R_{jl} - \delta^i_l R_{jk} + \delta^j_l R_{ik} - \delta^j_k R_{il}) + \frac{1}{6}R(\delta^i_k \delta_{jl} - \delta^i_l \delta_{jk}) \tag{39}$$

one can define a 2-form

$$W_{ij} = \frac{1}{2} \sum_{k,l} C^i_{jkl} e^k \wedge e^l. \tag{40}$$

With respect to the basis (5) the representation matrix W of (40) takes the block-diagonal form

$$W = \begin{pmatrix} W^+ & 0 \\ 0 & W^- \end{pmatrix} \tag{41}$$

where W^+ and W^- are 3×3 matrices representing the induced linear transformation of the invariant subspaces Λ^2_+ and Λ^2_- , respectively. If the constants c and d satisfy (34), the extended Taub–NUT metric (30) with $2 + cr > 0$ is conformally self-dual and one has [18]

$$W^+ = \frac{c}{2(a + br)(1 + cr/2)^2} W_0 \quad W^- = 0 \tag{42}$$

where W_0 is a diagonal matrix

$$W_0 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}. \tag{43}$$

For $2 + cr < 0$, the metric is conformally antiself-dual and the expressions of W^+ and W^- are interchanged.

Finally, the condition stated for a Stäckel–Killing tensor to be written as the square of a skew-symmetric tensor in the form (10) must be supplemented with equation (3), which defines a Killing–Yano tensor. To verify this last condition we shall use the Newman–Penrose formalism for Euclidean signature [24]. We introduce a tetrad which will be given as an isotropic complex dyad defined by the vectors l, m together with their complex conjugates subject to the normalization conditions

$$l_\mu \bar{l}^\mu = 1 \quad m_\mu \bar{m}^\mu = 1 \tag{44}$$

with all others vanishing and the metric is expressed in the form

$$ds^2 = l \otimes \bar{l} + \bar{l} \otimes l + m \otimes \bar{m} + \bar{m} \otimes m. \tag{45}$$

For a Stäckel–Killing tensor K with two distinct eigenvalues one can choose the tetrad in such a form that

$$K_{\mu\nu} = 2\lambda_1^2 l_{[\mu} \bar{l}_{\nu]} + 2\lambda_2^2 m_{[\mu} \bar{m}_{\nu]}. \tag{46}$$

The skew-symmetric tensor $f_{\mu\nu}$ which enters the decomposition (10) has the form

$$f_{\mu\nu} = 2\lambda_1 l_{[\mu} \bar{l}_{\nu]} + 2\lambda_2 m_{[\mu} \bar{m}_{\nu]}. \tag{47}$$

Again taking the example of the third component S_3 , the eigenvalues λ_1 and λ_2 are given by (35) and (37) and the tetrad (44) can be inferred from the eigenvectors (36) and (38) through a standard orthonormalization procedure. Finally, imposing equation (3), we find that (47) is a Killing–Yano tensor only if

$$c = \frac{2b}{a}. \tag{48}$$

With this constraint, together with (34), the extended metric (13) coincides, up to a constant factor, with the original Taub–NUT metric on setting $a/b = 4m$. Note that in equations (35)–(38) the constant α_1 is not fixed. In fact, the product of two Killing–Yano tensors $f' \cdot f''$ is invariant under the rescaling $f' \rightarrow \alpha f'$, $f'' \rightarrow \frac{1}{\alpha} f''$. Choosing adequately the normalization of the Killing–Yano tensors, for $\alpha_1 = -\frac{1}{4m}$ we recover precisely the original Taub–NUT decomposition (28) with $f' = f_i$ and $f'' = f_Y$ normalized as in (22) and (24).

4. Concluding remarks

The aim of this paper is to show that the extensions of the Taub–NUT geometry do not admit a Killing–Yano tensor, even if they possess Stäckel–Killing tensors.

This result is not unexpected. The conserved quantities $K_{i\mu\nu}$ which enter equation (28) are the components of the Runge–Lenz vector \vec{K} given in (26). In the original Taub–NUT case these components $K_{i\mu\nu}$ are related to the symmetrized products between the Killing–Yano tensors f_i (22) and f_Y (24). The three Killing–Yano tensors f_i transform as vectors under rotations generated by \vec{J} like the Runge–Lenz vector (32), while f_Y is a scalar.

The extended Taub–NUT metrics are not Ricci flat and, consequently, not hyper-Kähler. On the other hand, the existence of the Killing–Yano tensors f_i is correlated to the hyper-Kähler, self-dual structure of the metric.

The non-existence of the Killing–Yano tensors makes the study of ‘hidden’ symmetries more laborious in models of relativistic particles with spin involving anticommuting vectorial degrees of freedom. In general, the conserved quantities from the scalar case receive a spin contribution involving an even number of Grassmann variables ψ^μ . For example, starting with a Killing vector K_μ , the conserved quantity in the spinning case is

$$J(x, \dot{x}, \psi) = K^\mu \dot{x}_\mu + \frac{1}{2} i K_{[\mu;\nu]} \psi^\mu \psi^\nu. \tag{49}$$

The first term in the right-hand side is the conserved quantity in the scalar case, while the last term represents the contribution of the spin.

A ‘hidden’ symmetry is encapsulated in a Stäckel–Killing tensor of valence $r > 1$. The generalized Killing equations on spinning spaces including a Stäckel–Killing tensor are more involved. Unfortunately, it is not possible to write closed, analytic expressions of the solutions of these equations using directly the components of the Stäckel–Killing tensors. However, assuming that the Stäckel–Killing tensors can be written as symmetrized products of pairs of Killing–Yano tensors, the evaluation of the spin corrections is feasible [11, 12, 15, 16].

If the Killing–Yano tensors are missing, to take up the question of the existence of extra supersymmetries and the relation with the constants of motion we are forced to enlarge the approach to Killing equations (3) and (4). In fact, in [15], supersymmetries are shown to depend on the existence of a tensor field $f_{\mu\nu}$ satisfying equation (3) which will be referred to as the f -symbol. The general conditions for constants of motion were derived, and it was shown that one can have new supercharges which do not commute with the original supercharge $Q = \dot{x}_\mu \psi^\mu$ if one allows the f -symbols to have a symmetric part. It was shown that in this case the antisymmetric part does not satisfy the Killing–Yano condition (3). We would like to remark that the general conditions of [15] allow more possibilities than Killing–Yano tensors for the construction of supercharges.

In summary, we believe that the relation between the f -symbols and the Killing–Yano tensors could be fruitful and that it deserves further study. An analysis of the f -symbols in the generalized Taub–NUT geometry is underway [25].

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